

JACOB'S LADDERS, INTERACTIONS BETWEEN ζ -OSCILLATING SYSTEMS AND ζ -ANALOGUE OF AN ELEMENTARY TRIGONOMETRIC IDENTITY

JAN MOSER

ABSTRACT. In our previous papers, we have introduced within the theory of the Riemann zeta function the following notions: Jacob's ladders, oscillating systems, ζ -factorization, metamorphoses, ... In this paper we obtain ζ -analogue of an elementary trigonometric identity and other interactions between oscillating systems.

1. INTRODUCTION AND SURVEY OF NOTIONS WE HAVE INTRODUCED IN THE THEORY OF THE RIEMANN ZETA-FUNCTION

In our previous papers [1] – [7] we have introduced within the theory of the Riemann zeta-function following notions: Jacob's ladders (JL), ζ -oscillating systems (OS), factorization formula (FF), metamorphosis of the oscillating systems (M), Z_{ζ, Q^2} -transformation (ZT).

In the present paper we introduce the notion of interactions between oscillating systems (IOS).

The diagram of hierarchy of the previously mentioned notions is as follows:

$$JL \rightarrow OS \rightarrow FF \rightarrow \begin{cases} M \\ ZT \\ IOS \end{cases}$$

where IOS represents new level of our theory.

The main result obtained in this direction is the following set of the ζ -analogue of the elementary trigonometric identity $\cos^2 t + \sin^2 t = 1$:

$$\cos^2(\alpha_0^{2,2}) \prod_{r=1}^{k_2} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{2,2}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^2\right)} \right|^2 + \sin^2(\alpha_0^{1,1}) \prod_{r=1}^{k_1} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{1,1}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^1\right)} \right|^2 \sim 1, \quad L \rightarrow \infty.$$

Of course, we will obtain also other interactions between oscillating systems.

1.1. Let us remind that the Jacob's ladders

$$\varphi_1(t) = \frac{1}{2}\varphi(t)$$

we have introduced in [1] (see also [2]), where the function $\varphi(t)$ is an arbitrary solution to the nonlinear integral equation

$$\int_0^{\mu[x(T)]} Z^2(t) e^{-\frac{2}{x(T)}t} dt = \int_0^T Z^2(t) dt,$$

Key words and phrases. Riemann zeta-function.

where each admissible function $\mu(y)$ generates the solution

$$y = \varphi(T; \mu) = \varphi(T), \quad \mu(y) > 7y \ln y.$$

We call the function $\varphi_1(T)$ as Jacob's ladder as an analogue of the Jacob's dream in Chumash, Bereishis, 28:12.

Remark 1. By making use of these Jacob's ladders we have shown (see [1]) that the classical Hardy-Littlewood integral (1918)

$$\int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right|^2 dt$$

has - in addition to previously known Hardy-Littlewood expression (and other similar ones) possessing an unbounded error term at $T \rightarrow \infty$ - the following infinite set of almost exact representations

$$\begin{aligned} \int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right|^2 dt &= \varphi_1(t) \ln \{\varphi_1(t)\} + \\ &+ (c - \ln 2\pi) \varphi_1(t) + c_0 + \mathcal{O} \left(\frac{\ln T}{T} \right), \quad T \rightarrow \infty, \end{aligned}$$

where c is the Euler's constant and c_0 is the constant from the Titchmarsh-Kober-Atkinson formula.

1.2. Next, let us remind that we have introduced (see [7], (1.1)) the notion of the (ζ, Q^2) -oscillating system. At this place we give the definition in the following form.

Definition 1. The general oscillating system

$$[\zeta, Q^2, k], \quad k = 1, \dots, k_0, \quad k_0 \in \mathbb{N}$$

(with k_0 being arbitrary and fixed) is defined as follows

$$[\zeta, Q^2, k] = \prod_{r=1}^k \left| \frac{\zeta \left(\frac{1}{2} + ix_r \right)}{\zeta \left(\frac{1}{2} + iy_r \right)} \right|^2,$$

where

$$\begin{aligned} 0 &< T_0 < x_1 < x_2 < \dots < x_k, \\ T_0 &< y_1 < y_2 < \dots < y_k, \end{aligned}$$

and

$$\begin{aligned} (x_1, \dots, x_k), (y_1, \dots, y_k) &\in (T_0, +\infty)^k \\ x_r, y_r &\neq \gamma : \zeta \left(\frac{1}{2} + i\gamma \right) = 0, \quad r = 1, \dots, k \end{aligned}$$

for sufficiently big T_0 .

1.3. First of all we define the following class of admissible functions.

Definition 2. The symbol

$$(1.1) \quad f(t) \in \tilde{C}[T, T+U]$$

stands for the following

$$(1.2) \quad f(t) \in C[T, T+U] \wedge f(t) > 0$$

for

$$(1.3) \quad \begin{aligned} T &> T_0, \quad U \in (0, U_0], \\ U_0 &= o\left(\frac{T}{\ln T}\right), \quad T \rightarrow \infty. \end{aligned}$$

Remark 2. What concerns the last condition, see our paper [3], (7.1), (7.2).

Next, we have shown in [5], (4.3) – (4.18), (comp. [7], (2.1) – (2.7)) the following is true:

(A) there is a vector-operator \hat{H} acting on \tilde{C}

$$f(t) \mapsto \hat{H}f(t) = (\alpha_0, \alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k), \quad k = 1, \dots, k_0$$

where

$$(1.4) \quad \begin{aligned} \alpha_r &= \alpha_r(T, U, k; f), \quad r = 0, 1, \dots, k, \\ \beta_r &= \beta_r(T, U, k), \quad r = 1, \dots, k, \end{aligned}$$

(B) there is the factorization formula

$$(1.5) \quad \begin{aligned} f(t) &\longrightarrow \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + ix_r\right)}{\zeta\left(\frac{1}{2} + iy_r\right)} \right|^2 = \\ &= \left\{ 1 + \mathcal{O}\left(\frac{\ln \ln T}{\ln T}\right) \right\} \frac{H(T, U; f)}{f(\alpha_0)} \sim \\ &\sim \frac{H(T, U; f)}{f(\alpha_0)}, \quad T \rightarrow \infty, \end{aligned}$$

where

$$\begin{aligned} H(T, U; f) &= \frac{F(T+U; f) - F(T; f)}{U}, \\ F'_t(t; f) &= f(t). \end{aligned}$$

Now we give

Definition 3. Let

$$f(t) \in \tilde{C}[T, T+U].$$

Then the particular oscillating system

$$[\zeta, Q^2, k; f], \quad k = 1, \dots, k_0$$

is defined as follows

$$(1.6) \quad [\zeta, Q^2, k; f] = \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r\right)}{\zeta\left(\frac{1}{2} + i\beta_r\right)} \right|^2.$$

Remark 3. In this case we have (see (3.10), $c < 0.6$)

$$\begin{aligned} \alpha_{r+1} - \alpha_r &> 0.4 \times \frac{T}{\ln T}, \quad 0, 1, \dots, k-1, \\ \beta_{r+1} - \beta_r &> 0.4 \times \frac{T}{\ln T}, \quad r = 1, \dots, k-1, \end{aligned}$$

and consequently, the inequalities of Definition 1 are fulfilled.

1.4. We have used words *oscillating systems* in our Definitions 2 and 3. The main reason for this is in the spectral form of the Riemann-Siegel formula (see [5], (3.1) – (3.8))

$$\begin{aligned} Z(t) &= \sum_{n \leq \tau(x_r)} \frac{2}{\sqrt{n}} \cos\{t\omega_n(x_r) + \psi(x_r)\} + R(x_r), \\ \tau(x_r) &= \sqrt{\frac{x_r}{2\pi}}, \\ R(x_r) &= \mathcal{O}(x_r^{-1/4}), \\ t &\in [x_r, x_r + V], \quad V \in (0, x_r^{1/4}), \\ Z(t) &= e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right) \Rightarrow |Z(t)| = \left|\zeta\left(\frac{1}{2} + it\right)\right|, \end{aligned}$$

where the functions

$$\frac{2}{\sqrt{n}} \cos\{t\omega_n(x_r) + \psi(x_r)\}$$

are the Riemann's oscillators with:

(a) the amplitude

$$\frac{2}{\sqrt{n}},$$

(b) the incoherent local phase constant

$$\psi(x_r) = -\frac{x_r}{2} - \frac{\pi}{8},$$

(c) the nonsynchronized local times

$$t \in [x_r, x_r + V],$$

(d) local spectrum of cyclic frequencies

$$\{\omega_n(x_r)\}_{n \leq \tau(x_r)}, \quad \omega_n(x_r) = \ln \frac{\tau(x_r)}{n}.$$

Similar formulae take place for y_r .

Remark 4. We have, for example, (see Definition 1)

$$\begin{aligned} (1.7) \quad [\zeta, Q^2, k] &= \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + ix_r\right)}{\zeta\left(\frac{1}{2} + iy_r\right)} \right|^2 = \\ &= \prod_{r=1}^k \left| \frac{\sum_{n \leq \tau(x_r)} \frac{2}{\sqrt{n}} \cos\{x_r \omega_n(x_r) + \psi(x_r)\} + R(x_r)}{\sum_{n \leq \tau(y_r)} \frac{2}{\sqrt{n}} \cos\{y_r \omega_n(y_r) + \psi(y_r)\} + R(y_r)} \right|, \\ &k = 1, \dots, k_0. \end{aligned}$$

We see that (1.7) is quite a complicated function (we can take k_0 arbitrarily big).

Remark 5. The Riemann-Siegel formula (see [8], p. 60)

$$Z(t) = \sum_{n \leq \tau(t)} \frac{2}{\sqrt{n}} \cos\{\vartheta(t) - t \ln n\} + \mathcal{O}(t^{-1/4}),$$

where

$$Z(t) = e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right),$$

$$\vartheta(t) = -\frac{t}{2} \ln \pi + \operatorname{Im} \ln \Gamma\left(\frac{1}{4} + i\frac{t}{2}\right)$$

is the Riemann formula that has been restored (by Riemann's manuscripts) and published by C.L. Siegel.

Remark 6. Let us notice that by our opinion the Riemann-Siegel formula represents Riemann's fundamental contribution to the theory of oscillations (independently from the analytic number theory). Namely, the Riemann's oscillations are fated to describe of profound laws of our Universe.

2. NEW FORMULAE: INTERACTIONS OF OSCILLATING SYSTEMS AND, ESPECIALLY, ζ -ANALOGUE OF THE ELEMENTARY TRIGONOMETRIC IDENTITY

2.1. Let

$$(2.1) \quad \begin{aligned} f_1(t) &= \sin^2 t, \quad t \in [\pi L + \mu, \pi L + \mu + U], \\ L &\in \mathbb{N}, \quad U \in (0, U_0], \quad \mu \geq \mu_0 > 0, \\ 2\mu + U_0 &\leq \frac{\pi}{2} - \epsilon, \quad \epsilon > 0, \quad \pi L + \mu > T_0 \end{aligned}$$

(with ϵ, μ_0 being sufficiently small and fixed). Of course,

$$f_1(t) \in \tilde{C}[\pi L + \mu, \pi L + \mu + U].$$

Now, if we use our algorithm (comp. third part of this paper) to the function $f_1(t)$, then we obtain the following factorization formula

$$(2.2) \quad \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^1\right)}{\zeta\left(\frac{1}{2} + i\beta_r\right)} \right|^2 \sim \left\{ \frac{1}{2} - \frac{1}{2} \frac{\sin U}{U} \cos(2\mu + U) \right\} \frac{1}{\sin^2(\alpha_0^1)}, \quad L \rightarrow \infty,$$

where

$$\begin{aligned} \alpha_r^1 &= \alpha_r(U, \mu, L, k; \sin^2 t), \quad r = 0, 1, \dots, k, \\ \beta_r &= \beta_r(U, \mu, L, k), \quad r = 1, \dots, k, \\ \pi L + \mu &< \alpha_0^1 < \pi L + \mu + U \Rightarrow \mu < \alpha_0^1 - \pi L < \mu + U, \\ k &= 1, \dots, k_0. \end{aligned}$$

2.2. Next, we consider the function

$$(2.3) \quad f_2(t) = \cos^2 t, \quad t \in [\pi L + \mu, \pi L + \mu + U]$$

and obtain, by the similar way, the following factorization formula

$$(2.4) \quad \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^2\right)}{\zeta\left(\frac{1}{2} + i\beta_r\right)} \right|^2 \sim \left\{ \frac{1}{2} + \frac{1}{2} \frac{\sin U}{U} \cos(2\mu + U) \right\} \frac{1}{\cos^2(\alpha_0^2)}, \quad L \rightarrow \infty,$$

where

$$\begin{aligned} \alpha_r^2 &= \alpha_r(U, \mu, L, k; \cos^2 t), \quad r = 0, 1, \dots, k, \\ \mu &< \alpha_0^2 - \pi L < \mu + U, \quad k = 1, \dots, k_0. \end{aligned}$$

2.3. Consequently, we have obtained the following two sets

$$(2.5) \quad \begin{aligned} & \{[\zeta, Q^2, k_1; \sin^2 t]\}_{k_1=1}^{k_0}, \\ & \{[\zeta, Q^2, k_2; \cos^2 t]\}_{k_2=1}^{k_0} \end{aligned}$$

of particular oscillating systems.

Remark 7. We shall use the shortened phrase *oscillating systems* instead of *particular oscillating systems* in similar cases.

Now the following sets of new formulas is generated by two sets (2.5)

$$(2.6) \quad \begin{aligned} & \cos^2(\alpha_0^{2,2}) \prod_{r=1}^{k_2} \left| \frac{\zeta(\frac{1}{2} + i\alpha_r^{2,2})}{\zeta(\frac{1}{2} + i\beta_r^2)} \right|^2 + \sin^2(\alpha_0^{1,1}) \prod_{r=1}^{k_1} \left| \frac{\zeta(\frac{1}{2} + i\alpha_r^{1,1})}{\zeta(\frac{1}{2} + i\beta_r^1)} \right|^2 \sim 1, \quad L \rightarrow \infty, \\ & k = 1, \dots, k_0, \end{aligned}$$

or, for example,

$$(2.7) \quad \begin{aligned} & \prod_{r=1}^{k_2} \left| \frac{\zeta(\frac{1}{2} + i\alpha_r^{2,2})}{\zeta(\frac{1}{2} + i\beta_r^2)} \right|^2 \sim \\ & \sim \frac{1}{\cos^2(\alpha_0^{2,2})} - \frac{\sin^2(\alpha_0^{1,1})}{\cos^2(\alpha_0^{2,2})} \prod_{r=1}^{k_1} \left| \frac{\zeta(\frac{1}{2} + i\alpha_r^{1,1})}{\zeta(\frac{1}{2} + i\beta_r^1)} \right|^2, \quad L \rightarrow \infty \end{aligned}$$

(and second similar formula), where

$$\begin{aligned} \alpha_0^{1,1} &= \alpha_0^1(U, \mu, L, k_1; \sin^2 t), \dots \\ \beta_r^1 &= \beta_r(U, \mu, L, k_1), \\ \alpha_0^{2,2} &= \alpha_0^2(U, \mu, L, k_2; \cos^2 t), \dots \\ \beta_r^2 &= \beta_r(U, \mu, L, k_2). \end{aligned}$$

Remark 8. We call the formula (2.6) as the ζ -analogue of the elementary trigonometric formula

$$\sin^2 t + \cos^2 t = 1.$$

Remark 9. Of course, we have within (2.6) huge number

$$(k_0)^2$$

of distinct formulas.

2.4. It follows from (2.6), (2.7) that the corresponding oscillating systems (see (2.5))

$$(2.8) \quad [\zeta, Q^2, k_1; \sin^2 t], [\zeta, Q^2, k_2; \cos^2 t]$$

are functionally depending systems. That is, if all parameters

$$U, \mu, L, k_1, k_2$$

are fixed, then the corresponding values of the oscillating systems are linearly connected (in the asymptotic sense).

Definition 4. We shall call the above mentioned functional dependence of the oscillating systems (2.8) as interaction between them and we shall denote this by the following diagram

$$(2.9) \quad \begin{aligned} [\zeta, Q^2, k_1; \sin^2 t] &\longleftrightarrow [\zeta, Q^2, k_2; \cos^2 t], \\ 1 \leq k_1, k_2 &\leq k_0. \end{aligned}$$

Consequently, this paper is devoted to study of interactions between oscillating systems of the second order (as (2.9)), and also to the third order systems, that is if

$$[\zeta, Q^2, k_l, f_l] \longrightarrow (k_l, f_l), \quad l = 1, 2, 3.$$

In this sense, we will study the following diagrams

$$(k_1, f_1) \longleftrightarrow (k_2, f_2), \quad (k_1, f_1) \longleftrightarrow (k_2, f_2) \longleftrightarrow (k_3, f_3) \longleftrightarrow (k_1, f_1)$$

for

$$1 \leq k_1, k_2, k_3 \leq k_0.$$

3. SHORT SURVEY OF OUR ALGORITHM FOR GENERATING THE FACTORIZATION FORMULAE

3.1. If

$$f(t) \in \tilde{C}[T, T+U]$$

then the formula

$$(3.1) \quad \begin{aligned} \frac{1}{U} \int_T^{T+U} f(t) dt &= H(T, U; f) > 0, \quad U \in (0, U_0], \\ H(T, U; f) &= \frac{F(T+U; f) - F(T; f)}{U}, \\ F'_t(t; f) &= f(t) \end{aligned}$$

holds true.

3.2. Next, let us remind the we have introduced the following new type of integral in the theory of the Riemann zeta-function (see [2], (9.5), [3], (7.1), (7.2)): if

$$f(t) \in \tilde{C}[T, T+U]$$

then

$$(3.2) \quad \begin{aligned} &\int_T^{\widehat{T+U}^k} f[\varphi_1^k(t)] \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^r(t)] dt = \\ &= \int_T^{T+U} f(t) dt \Rightarrow \\ &\Rightarrow \int_T^{\widehat{T+U}^k} \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^r(t)] dt = U; \quad f(t) = 1, \end{aligned}$$

where (see [2], (9.1), (9.2))

$$(3.3) \quad \begin{aligned} \tilde{Z}^2(t) &= \frac{|\zeta(\frac{1}{2} + it)|}{\omega(t)}, \\ \omega(t) &= \left\{ 1 + \mathcal{O}\left(\frac{\ln \ln t}{\ln t}\right) \right\} \ln t, \end{aligned}$$

and (see [3], (6.1), (6.4))

$$[\overbrace{T, T+U}^k]; [T, T+U] \prec [\overbrace{T, T+U}^1] \prec \dots \prec [\overbrace{T, T+U}^k]$$

is the k -th reverse iteration of the segment $[T, T+U]$ where (see [3], (6.1))

$$\varphi_1(T) = \overbrace{T}^k, \overbrace{T}^0 = T.$$

3.3. Next, we obtain from (3.2) by the mean-value theorem and (3.1) (comp. [5], (4.5) – (4.7))

$$(3.4) \quad \begin{aligned} f(\alpha_0) \prod_{r=1}^k \tilde{Z}^2(\alpha_r) &= \frac{U}{\overbrace{T+U-T}^k} H(T, U; f), \\ \alpha_r &= \varphi_1^{k-r}(d), \quad r = 0, 1, \dots, k, \\ d &= d(T, U, k; f), \quad d \in \{d\}, \end{aligned}$$

and, by the similar way, (comp. [5], (4.16), (4.17))

$$(3.5) \quad \begin{aligned} \prod_{r=1}^k \tilde{Z}^2(\beta_r) &= \frac{U}{\overbrace{T+U-T}^k}, \\ \beta_r &= \varphi_1^{k-r}(e), \quad r = 1, \dots, k, \\ e &= e(T, U, k; f), \quad e \in \{e\}, \end{aligned}$$

where

$$\{d\}, \{e\}$$

are sets of the abscises of the corresponding mean-values (3.4) and (3.5).

Remark 10. If we choose

(a) only one

$$d \in \{d\},$$

(b) only one

$$e \in \{e\},$$

then we obtain (see (3.4), (3.5)) the corresponding vector-valued function

$$(3.6) \quad (\alpha_0, \alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k), \quad k = 1, \dots, k_0,$$

i.e. we have k_0 corresponding vector-valued functions. Of course, these sets are defined for every admissible and fixed

$$T, U, k; f(t).$$

Now, we give the following

Definition 5. By making use the mean-value theorem three times (see (3.1), (3.4) and (3.5)) we define the vector-operator as follows

$$(3.6) \quad \begin{aligned} \forall f(t) \in \tilde{C}[T, T+U] &\longrightarrow \hat{H}f(t) = \\ &= (\alpha_0, \alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k), \quad k = 1, \dots, k_0. \end{aligned}$$

Remark 11. We notice explicitly that the operator \hat{H} represents new type of operator since its definition is based on the function

$$\zeta\left(\frac{1}{2} + it\right)$$

as well as on the Jacob's ladder (see (3.4) and (3.5)).

3.4. Further we obtain, eliminating the factor

$$\frac{U}{\overbrace{T+U}^k - T^k}$$

from (3.4) and (3.5), the following formula

$$(3.7) \quad \prod_{r=1}^k \frac{\tilde{Z}^2(\alpha_r)}{\tilde{Z}^2(\beta_r)} = \frac{H(T, U; f)}{f(\alpha_0)}, \quad T \rightarrow \infty.$$

Remark 12. We call formula (3.7) as *exact factorization formula*.

Next, we obtain from (3.7) by (3.3), (comp. [5], (4.11) – (4.14), [7], (2.7)) the following formula

$$(3.8) \quad \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r\right)}{\zeta\left(\frac{1}{2} + i\beta_r\right)} \right|^2 = \left\{ 1 + \mathcal{O}\left(\frac{\ln \ln T}{\ln T}\right) \right\} \frac{H(T, U; f)}{f(\alpha_0)} \sim \frac{H(T, U; f)}{f(\alpha_0)}, \quad T \rightarrow \infty.$$

Remark 13. We call formula (3.8) as *asymptotic factorization formula*.

3.5. In this paragraph, we present some properties of the vector operator \hat{H} (properties of the corresponding vector-valued function in (3.6)).

Property 1. Since (see (1.2), (1.3))

$$f(t) \in \tilde{C}[T, T+U] \Rightarrow f(t) > 0,$$

then (see (3.1), (3.3) – (3.5))

$$(3.9) \quad \alpha_r \neq \gamma, \beta_r \neq \gamma : \zeta\left(\frac{1}{2} + it\right) \Big|_{t=\gamma} = 0, \quad r = 1, \dots, k.$$

Remark 14. Consequently, the set

$$\{\gamma\}, \quad \gamma > T_0$$

is the exceptional one for the last $2k$ components

$$(\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k)$$

of the vector-valued function (3.6).

Property 2. Next, the following inclusions holds true (comp. [7], (6.3))

$$\alpha_0 \in (T, T+U), \quad \alpha_r, \beta_r \in (\overbrace{T, T+U}^r), \quad r = 1, \dots, k.$$

Property 3.

$$(3.10) \quad \begin{aligned} \alpha_{r+1} - \alpha_r &\sim (1-c)\pi(T), \quad r = 0, 1, \dots, k-1, \\ \beta_{r+1} - \beta_r &\sim (1-c)\pi(T), \quad r = 1, \dots, k-1, \end{aligned}$$

where

$$\pi(T) \sim \frac{T}{\ln T}, \quad T \rightarrow \infty$$

is the prime-counting function and c is the Euler's constant (comp. [5], (2.8)).

Remark 15. Jacob's ladder $\varphi_1(T)$ can be viewed by our formula (see [1], (6.2))

$$T - \varphi_1(T) \sim (1-c)\pi(T)$$

as an asymptotic complementary function to the function

$$(1-c)\pi(T)$$

in the following sense

$$\varphi_1(T) + (1-c)\pi(T) \sim T, \quad T \rightarrow \infty.$$

Remark 16. The asymptotic behavior of the sequences

$$(3.11) \quad \{\alpha_r\}_{r=0}^k, \quad \{\beta_r\}_{r=1}^k$$

is by (3.10) as follows (see [5], Remark 2): if $T \rightarrow \infty$ then the points of every sequence (3.11) recede unboundedly each from other and all together recede to infinity. Hence, at $T \rightarrow \infty$ each sequence in (3.11) behaves as one-dimensional Friedmann-Hubble universe.

4. FIRST SETS OF LEMMAS

4.1. The following lemma holds true.

Lemma 1. The function

$$(4.1) \quad f_1(t) = \sin^2 t \in \tilde{C}[\pi L + \mu, \pi L + \mu + U], \quad L \in \mathbb{N}, \quad \pi L > T_0$$

corresponds with the following factorization formula

$$(4.2) \quad \begin{aligned} \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^1\right)}{\zeta\left(\frac{1}{2} + i\beta_r\right)} \right|^2 &\sim \\ &\sim \left\{ \frac{1}{2} - \frac{1}{2} \frac{\sin U}{U} \cos(2\mu + U) \right\} \frac{1}{\sin^2(\alpha_0^1)}, \quad L \rightarrow \infty, \end{aligned}$$

where

$$(4.3) \quad \begin{aligned} \alpha_r^1 &= \alpha_r(U, \mu, L, k; \sin^2 t), \quad r = 0, 1, \dots, k, \\ \beta_r &= \beta_r(U, \mu, L, k), \quad r = 1, \dots, k, \\ \pi L + \mu &< \alpha_0^1 < \pi L + \mu + U \Rightarrow \mu < \alpha_0^1 - \pi L < \mu + U, \\ k &= 1, \dots, k_0. \end{aligned}$$

Proof. Since

$$\int \sin^2 t dt = \frac{t}{2} - \frac{1}{4} \sin 2t + C,$$

then

$$\int_{\pi L + \mu}^{\pi L + \mu + U} \sin^2 t dt = \frac{U}{2} - \frac{1}{2} \sin U \cos(2\mu + U),$$

and, of course,

$$(4.4) \quad \begin{aligned} & \frac{1}{U} \int_{\pi L + \mu}^{\pi L + \mu + U} \sin^2 t dt = \\ & = \frac{1}{2} - \frac{1}{2} \frac{\sin U}{U} \cos(2\mu + U). \end{aligned}$$

Consequently, we obtain (4.2) from (4.4) by our algorithm (see third part of this paper). \square

4.2. By the similar way we obtain the following

Lemma 2. The function

$$(4.5) \quad f_2(t) = \cos^2 t \in \tilde{C}[\pi L + \mu, \pi L + \mu + U], \quad L \in \mathbb{N}, \quad \pi L > T_0$$

corresponds with the following factorization formula

$$(4.6) \quad \begin{aligned} & \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^2\right)}{\zeta\left(\frac{1}{2} + i\beta_r\right)} \right|^2 \sim \\ & \sim \left\{ \frac{1}{2} + \frac{1}{2} \frac{\sin U}{U} \cos(2\mu + U) \right\} \frac{1}{\cos^2(\alpha_0^2)}, \quad L \rightarrow \infty, \end{aligned}$$

where

$$(4.7) \quad \begin{aligned} & \alpha_r^2 = \alpha_r(U, \mu, L, k; \cos^2 t), \quad r = 0, 1, \dots, k, \\ & \dots, \\ & \mu < \alpha_0^2 - \pi L < \mu + U, \\ & k = 1, \dots, k_0. \end{aligned}$$

4.3. Now we obtain the following

Lemma 3. The function

$$(4.8) \quad f_3(t) = \frac{1}{\cos^2 t} \in \tilde{C}[\pi L + \mu, \pi L + \mu + U], \quad L \in \mathbb{N}, \quad \pi L > T_0$$

corresponds with the following factorization formula

$$(4.9) \quad \begin{aligned} & \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^3\right)}{\zeta\left(\frac{1}{2} + i\beta_r\right)} \right|^2 \sim \\ & \sim \frac{\sin U}{U} \frac{\cos^2(\alpha_0^3)}{\cos(\mu + U) \cos \mu}, \quad L \rightarrow \infty, \end{aligned}$$

where

$$(4.10) \quad \begin{aligned} & \alpha_r^3 = \alpha_r(U, \mu, L, k; 1/\cos^2 t), \quad r = 0, 1, \dots, k, \\ & \dots, \\ & \mu < \alpha_0^3 - \pi L < \mu + U, \\ & k = 1, \dots, k_0. \end{aligned}$$

Proof. Since

$$\int \frac{dt}{\cos^2 t} = \tan t + C,$$

then

$$\int_{\pi L + \mu}^{\pi L + \mu + U} \frac{dt}{\cos^2 t} = \tan(\mu + U) - \tan \mu = \frac{\sin U}{\cos(\mu + U) \cos \mu},$$

and, of course,

$$(4.11) \quad \frac{1}{U} \int_{\pi L + \mu}^{\pi L + \mu + U} \frac{dt}{\cos^2 t} = \frac{\sin U}{U} \frac{1}{\cos(\mu + U) \cos \mu}.$$

Consequently, the formula (4.9) follows from (4.11) by our algorithm. \square

5. METAMORPHOSIS AS THE FIRST INTERPRETATION OF THE FACTORIZATION FORMULA

5.1. First of all, we have (see (3.8)) that

$$(5.1) \quad \begin{aligned} \forall f(t) \in \tilde{C}[T, T + U] &\rightarrow \\ &\rightarrow \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r\right)}{\zeta\left(\frac{1}{2} + i\beta_r\right)} \right|^2 \sim \\ &\sim \frac{H(T, U; f)}{f(\alpha_0)}, \quad T \rightarrow \infty, \quad k = 1, \dots, k_0, \\ \alpha_r &= \alpha_r(T, U, k; f), \quad r = 0, 1, \dots, k, \\ \beta_r &= \beta_r(T, U, k; f), \quad r = 1, \dots, k, \end{aligned}$$

i.e. a factorization formula corresponds to every admissible function $f(t)$.

5.2. Next, let us remind the general oscillating system (see Definition 1)

$$(5.2) \quad [\zeta, Q^2, k; f] = \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + ix_r\right)}{\zeta\left(\frac{1}{2} + iy_r\right)} \right|^2, \quad k = 1, \dots, k_0,$$

where

$$\begin{aligned} T_0 &< x_1 < x_2 < \dots < x_k, \\ T_0 &< y_1 < y_2 < \dots < y_k, \\ (x_1, \dots, x_k), (y_1, \dots, y_k) &\in (T_0, +\infty)^k. \end{aligned}$$

In connection with (5.1) and (5.2) we give the following.

Remark 17. We have introduced:

- (a) the notion of metamorphoses of an oscillating multiform in our paper [4],
- (b) the notion of metamorphoses of a quotient of two oscillating multiforms in our paper [5].

5.3. The mechanism of metamorphoses is as follows. If we get, by random sampling, such points

$$(5.3) \quad (x_1, \dots, x_k), (y_1, \dots, y_k)$$

that

$$(5.4) \quad (x_1, \dots, x_k) = (\alpha_1, \dots, \alpha_k); \quad (y_1, \dots, y_k) = (\beta_1, \dots, \beta_k).$$

Then - at the points (5.3) obeying property (5.4) - the general oscillating system (5.2) changes its old form (chrysalis) into the new form (see (5.1))

$$\sim \frac{H(T, U; f)}{f(\alpha_0)}$$

(butterfly), and the last is controlled by the function α_0 .

6. \mathcal{Z}_{ζ, Q^2} -TRANSFORMATION AS THE SECOND INTERPRETATION OF THE FACTORIZATION FORMULA

6.1. In [7] we have introduced notion of \mathcal{Z}_{ζ, Q^2} -transformation. Namely, following transformation corresponds to the factorization formula (5.1): if

$$f(t) \in \tilde{C}[T, T + U],$$

then

$$(6.1) \quad \left(\begin{array}{c} f(t) \\ t \in [T, T + U] \end{array} \right) \xrightarrow{\mathcal{Z}_{\zeta, Q^2}} \left(\begin{array}{c} \frac{H(T, U; f)}{f(\alpha_0)} \\ U \in (0, U_0] \end{array} \right), \quad T \rightarrow \infty.$$

6.2. We may assume that the interpretation of (6.1) is especially effective for such the signals (or pulses) that appear in the theory of communication. In this direction we have introduced in our work [7] the following.

Definition 6. The \mathcal{Z}_{ζ, Q^2} -transformation we call as \mathcal{Z}_{ζ, Q^2} -device with its input and output.

In our paper [7] we have obtained the following main law for the class of power-signals (pulses): in this case

$$t^\Delta \in \tilde{C}[T, T + U], \quad \forall \Delta \in \mathbb{R}, \quad U \in (0, U_0], \\ U_0 < 1, \quad \forall L > L_0(\Delta), \quad L \in \mathbb{N}$$

we have (see (6.1))

$$(6.2) \quad \left(\begin{array}{c} t^\Delta \\ t \in [L, L + U] \end{array} \right) \xrightarrow{\mathcal{Z}_{\zeta, Q^2}} \left(\begin{array}{c} 1 \\ U \in (0, U_0] \end{array} \right), \quad L \rightarrow \infty,$$

where

$$\frac{H(L, U; f)}{(\alpha_0)^\Delta} \sim 1.$$

Remark 18. Every admissible power-signal (pulse)

$$\left(\begin{array}{c} t^\Delta \\ t \in [L, L + U] \end{array} \right)$$

on the input of the \mathcal{Z}_{ζ, Q^2} -device is transformed into telegraphic signal (pulse) that is into the unit rectangular signal (pulse). For example,

$$(6.3) \quad \left(\begin{array}{c} t^{1000} \\ t \in [L, L + U] \end{array} \right) \xrightarrow{\mathcal{Z}_{\zeta, Q^2}} \left(\begin{array}{c} 1 \\ U \in (0, U_0] \end{array} \right), \quad L \rightarrow \infty,$$

$$(6.4) \quad \left(\begin{array}{c} t^{-1000} \\ t \in [L, L + U] \end{array} \right) \xrightarrow{\mathcal{Z}_{\zeta, Q^2}} \left(\begin{array}{c} 1 \\ U \in (0, U_0] \end{array} \right), \quad L \rightarrow \infty.$$

Remark 19. We see that the unbounded signal (pulse) in (6.3) as well as the negligible signal (pulse) in (6.4) are both transformed by \mathcal{Z}_{ζ, Q^2} -device into the telegraphic signal (pulse).

7. THE ζ -ANALOGUE OF THE ELEMENTARY TRIGONOMETRIC IDENTITY AS THE FIRST EXAMPLE OF THE INTERACTIONS BETWEEN OSCILLATING SYSTEMS

7.1. Since all the coefficient-functions in the formulae (4.2), (4.6) are bounded and $\neq 0$ (in the conditions of (4.1)), then by eliminating the member

$$\frac{\sin U}{U} \cos(\mu + U),$$

we obtain the following.

Theorem 1. If the assumptions in (4.1) are fulfilled then

$$(7.1) \quad \cos^2(\alpha_0^{2,2}) \prod_{r=1}^{k_2} \left| \frac{\zeta(\frac{1}{2} + i\alpha_r^{2,2})}{\zeta(\frac{1}{2} + i\beta_r^2)} \right|^2 + \sin^2(\alpha_0^{1,1}) \prod_{r=1}^{k_1} \left| \frac{\zeta(\frac{1}{2} + i\alpha_r^{1,1})}{\zeta(\frac{1}{2} + i\beta_r^1)} \right|^2 \sim 1,$$

$$L \rightarrow \infty, \quad 1 \leq k \leq k_1, k_2 \leq k_0,$$

where

$$(7.2) \quad \begin{aligned} \alpha_0^{1,1} &= \alpha_0^1(U, \mu, L, k_1; \sin^2 t), \dots \\ \beta_r^1 &= \beta_r(U, \mu, L, k_1), \\ \alpha_0^{2,2} &= \alpha_0^2(U, \mu, L, k_2; \cos^2 t), \dots \\ \beta_r^2 &= \beta_r(U, \mu, L, k_2). \end{aligned}$$

Remark 20. We call the formula (7.1) as the ζ -analogue of the elementary trigonometric identity

$$\cos^2 t + \sin^2 t = 1.$$

Of course, the formula (7.1) denotes general element of the set

$$(k_0)^2$$

distinct formulas for every admissible and fixed L and for every fixed segment

$$[\pi L_\mu, \pi L + \mu + U].$$

Remark 21. Further, the formula (7.1) expresses the asymptotic dependence of every pair of oscillating systems

$$[\zeta, Q^2, k_1; \sin^2 t], [\zeta, Q^2, k_2, \cos^2 t], \quad 1 \leq k_1, k_2 \leq k_0$$

(comp. third part of this paper).

7.2. Now, we give explicitly (comp. (2.7) and Definition 4) the following (see (7.1)).

Corollary 1.

$$(7.3) \quad \begin{aligned} & \prod_{r=1}^{k_2} \left| \frac{\zeta(\frac{1}{2} + i\alpha_r^{2,2})}{\zeta(\frac{1}{2} + i\beta_r^2)} \right|^2 \sim \\ & \sim \frac{1}{\cos^2(\alpha_0^{2,2})} - \frac{\sin^2(\alpha_0^{1,1})}{\cos^2(\alpha_0^{2,2})} \prod_{r=1}^{k_1} \left| \frac{\zeta(\frac{1}{2} + i\alpha_r^{1,1})}{\zeta(\frac{1}{2} + i\beta_r^1)} \right|^2, \quad L \rightarrow \infty, \end{aligned}$$

$$\begin{aligned}
(7.4) \quad & \prod_{r=1}^{k_1} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{1,1}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^1\right)} \right|^2 \sim \\
& \sim \frac{1}{\sin^2(\alpha_0^{1,1})} - \frac{\cos^2(\alpha_0^{2,2})}{\sin^2(\alpha_0^{1,1})} \prod_{r=1}^{k_2} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{2,2}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^2\right)} \right|^2, \quad L \rightarrow \infty.
\end{aligned}$$

Let us remind (comp. (2.5)) that in our case, we have two sets of oscillating systems

$$\begin{aligned}
(7.5) \quad & \{[\zeta, Q^2, k_1; f_1]\}_{k_1=1}^{k_0}, \quad f_1 = f_1(t) = \sin^2 t, \\
& \{[\zeta, Q^2, k_2; f_2]\}_{k_2=1}^{k_0}, \quad f_2 = f_2(t) = \cos^2 t.
\end{aligned}$$

Remark 22. By the formulas (7.3) and (7.4) is expressed the property that we call (see Definition 4) as the interaction between two corresponding oscillating systems from distinct sets (7.5). We use the following diagram

$$(7.6) \quad [\zeta, Q^2, k_1; f_1] \longleftrightarrow [\zeta, Q^2, k_2, f_2]$$

to denote mentioned interaction.

8. THE SECOND CASE: INTERACTIONS BETWEEN THREE OSCILLATING SYSTEMS

8.1. Since (see (4.2), (4.6))

$$\begin{aligned}
(8.1) \quad & \frac{\cos^2(\alpha_0^{2,2})}{\cos(2\mu + U)} \prod_{r=1}^{k_2} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{2,2}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^2\right)} \right|^2 - \frac{\sin^2(\alpha_0^{1,1})}{\cos(2\mu + U)} \prod_{r=1}^{k_1} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{1,1}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^1\right)} \right|^2 \sim \\
& \sim \frac{\sin U}{U}, \quad L \rightarrow \infty
\end{aligned}$$

then we obtain (see (4.9)) the following

Theorem 2. If the assumptions of (4.1) are fulfilled then

$$\begin{aligned}
(8.2) \quad & \prod_{r=1}^{k_3} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{3,3}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^3\right)} \right|^2 \sim \\
& \sim \frac{\cos^2(\alpha_0^{2,2}) \cos^2(\alpha_0^{3,3})}{\cos(2\mu + U) \cos(\mu + U) \cos \mu} \prod_{r=1}^{k_2} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{2,2}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^2\right)} \right|^2 - \\
& - \frac{\sin^2(\alpha_0^{1,1}) \cos^2(\alpha_0^{3,3})}{\cos(2\mu + U) \cos(\mu + U) \cos \mu} \prod_{r=1}^{k_1} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{1,1}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^1\right)} \right|^2, \quad L \rightarrow \infty,
\end{aligned}$$

where (comp. (4.10))

$$\begin{aligned}
(8.3) \quad & \alpha_0^{3,3} = \alpha_0^3(U, \mu, L, k_3, f_3), \dots \\
& \beta_r^3 = \beta_r(U, \mu, L, k_3), \\
& f_3 = f_3(t) = \frac{1}{\cos^2 t}
\end{aligned}$$

and for the symbols

$$\alpha_0^{2,2}, \alpha_0^{1,1}, \dots$$

see (7.2).

8.2. Next, we give, to be complete, the following

Corollary 2.

$$(8.4) \quad \prod_{r=1}^{k_2} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{2,2}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^2\right)} \right|^2 \sim \\ \sim \frac{\sin^2(\alpha_0^{1,1})}{\cos^2(\alpha_0^{2,2})} \prod_{r=1}^{k_1} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{1,1}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^1\right)} \right|^2 + \\ + \frac{\cos(2\mu + U) \cos(\mu + U) \cos \mu}{\cos^2(\alpha_0^{2,2}) \cos^2(\alpha_0^{3,3})} \prod_{r=1}^{k_3} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{3,3}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^3\right)} \right|^2, \quad L \rightarrow \infty,$$

$$(8.5) \quad \prod_{r=1}^{k_1} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{1,1}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^1\right)} \right|^2 \sim \\ \sim \frac{\cos^2(\alpha_0^{2,2})}{\sin^2(\alpha_0^{1,1})} \prod_{r=1}^{k_2} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{2,2}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^2\right)} \right|^2 - \\ - \frac{\cos(2\mu + U) \cos(\mu + U) \cos \mu}{\sin^2(\alpha_0^{1,1}) \cos^2(\alpha_0^{3,3})} \prod_{r=1}^{k_3} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{3,3}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^3\right)} \right|^2, \quad L \rightarrow \infty.$$

8.3. Now, we have the following three sets of oscillating systems

$$(8.6) \quad \begin{aligned} & \{[\zeta, Q^2, k_1; f_1]\}_{k_1=1}^{k_0}, \quad f_1 = \sin^2 t, \\ & \{[\zeta, Q^2, k_2; f_2]\}_{k_2=1}^{k_0}, \quad f_2 = \cos^2 t, \\ & \{[\zeta, Q^2, k_3; f_3]\}_{k_3=1}^{k_0}, \quad f_3 = \frac{1}{\cos^2 t}. \end{aligned}$$

Remark 23. Consequently, new set of interactions corresponds to our three sets of oscillating systems (8.6), namely we have the third order diagram

$$[\zeta, Q^2, k_1; f_1] \longleftrightarrow [\zeta, Q^2, k_2, f_2] \longleftrightarrow [\zeta, Q^2, k_3, f_3] \longleftrightarrow [\zeta, Q^2, k_1, f_1].$$

Remark 24. Now we can expect there are diagrams of the order 4, 5, and so on.

9. ON NEW TYPE OF FACTORIZATION FORMULA GENERATED BY THE SECOND-LEVEL ELIMINATION

9.1. Since

$$\int_{2\pi L + \mu}^{2\pi L + \mu + U} \cos t dt = 2 \sin \frac{U}{2} \cos \left(\mu + \frac{U}{2} \right)$$

then

$$\frac{1}{U} \int_{2\pi L + \mu}^{2\pi L + \mu + U} \cos t dt = \frac{2}{U} \sin \frac{U}{2} \cos \left(\mu + \frac{U}{2} \right),$$

and we obtain by our algorithm the following

Lemma 4. Let

$$(9.1) \quad \begin{aligned} & f_4(t) = \cos t, \quad t \in [2\pi L + \mu, 2\pi L + \mu + U], \\ & U > 0, \quad \mu \geq \mu_0 > 0, \quad U \in (0, U_0], \quad \mu + \frac{U_0}{2} \leq \frac{\pi}{2} - \epsilon, \end{aligned}$$

where, of course,

$$f_4(t) \in \tilde{C}[2\pi L + \mu, 2\pi L + \mu + U].$$

Then the following factorization formula holds true

$$(9.2) \quad \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^4\right)}{\zeta\left(\frac{1}{2} + i\beta_r\right)} \right|^2 \sim \frac{2}{U} \sin \frac{U}{2} \frac{\cos\left(\mu + \frac{U}{2}\right)}{\cos(\alpha_0^4)}, \quad L \rightarrow \infty,$$

where

$$(9.3) \quad \begin{aligned} \alpha_0^4 &= \alpha_0(U, \mu, L, k; f_4), \dots \\ \beta_r &= \beta_r(U, \mu, L, k), \\ 2\pi L + \mu &< \alpha_0^4 < 2\pi L + \mu + U \Rightarrow \mu < \alpha_0^4 - 2\pi L < \mu + U, \end{aligned}$$

(comp. (9.1)).

9.2. Next, we obtain by the similar way the following

Lemma 5. Let

$$f_5(t) = \cos t, \quad t \in [2\pi L + \mu, 2\pi L + \mu + U],$$

(under the same assumptions as in (9.1)). Then the following factorization formula holds true

$$(9.4) \quad \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^5\right)}{\zeta\left(\frac{1}{2} + i\beta_r\right)} \right|^2 \sim \frac{2}{U} \sin \frac{U}{2} \frac{\cos\left(\mu + \frac{U}{2}\right)}{\sin(\alpha_0^5)}, \quad L \rightarrow \infty,$$

where

$$(9.5) \quad \begin{aligned} \alpha_0^5 &= \alpha_0(U, \mu, L, k; f_5), \dots \\ \beta_r &= \beta_r(U, \mu, L, k), \end{aligned}$$

(comp. (7.3)).

9.3. Since the right-hand side of formulae (9.2) and (9.4) are bounded and nonzero (see the assumptions in (9.1)), then we have the following.

Lemma 6. Under the assumptions (9.1) we have the following interaction formula

$$(9.6) \quad \begin{aligned} &\prod_{r=1}^{k_2} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{5,2}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^2\right)} \right|^2 \sim \\ &\sim \tan\left(\mu + \frac{U}{2}\right) \frac{\cos(\alpha_0^{4,1})}{\sin(\alpha_0^{5,2})} \prod_{r=1}^{k_1} \left| \frac{\zeta\left(\frac{1}{2} + i\alpha_r^{4,1}\right)}{\zeta\left(\frac{1}{2} + i\beta_r^1\right)} \right|^2, \quad L \rightarrow \infty, \end{aligned}$$

where

$$(9.7) \quad \begin{aligned} \alpha_0^{5,2} &= \alpha_0^5(U, \mu, L, k_2; f_5), \dots \\ \beta_r^2 &= \beta_r(U, \mu, L, k_2), \\ \alpha_0^{4,1} &= \alpha_0^4(U, \mu, L, k_1; f_4), \dots \\ \beta_r^1 &= \beta_r(U, \mu, L, k_1), \\ 1 &\leq k_1, k_2 \leq k_0. \end{aligned}$$

9.4. Now, in the case

$$(9.8) \quad k_1 = k_2 = k$$

it is true that (see (9.3), (9.7))

$$(9.9) \quad \beta_r^2 = \beta_r^1 = \beta_r, \quad r = 1, \dots, k_0.$$

Next, we have by (9.9)

$$\prod_{r=1}^{k_2} \left| \zeta \left(\frac{1}{2} + i\beta_r^2 \right) \right|^2 = \prod_{r=1}^{k_1} \left| \zeta \left(\frac{1}{2} + i\beta_r^1 \right) \right|^2 > 0$$

(comp. also (1.4), (1.5)) and, consequently,

$$(9.10) \quad \frac{\prod_{r=1}^k \left| \frac{\zeta(\frac{1}{2} + i\alpha_r^5)}{\zeta(\frac{1}{2} + i\beta_r)} \right|^2}{\prod_{r=1}^k \left| \frac{\zeta(\frac{1}{2} + i\alpha_r^4)}{\zeta(\frac{1}{2} + i\beta_r)} \right|^2} = \prod_{r=1}^k \left| \frac{\zeta(\frac{1}{2} + i\alpha_r^5)}{\zeta(\frac{1}{2} + i\alpha_r^4)} \right|^2.$$

Now, the following theorem holds true.

Theorem 3. Under the assumptions of (9.1) we have the following (new) type of interaction formula

$$(9.11) \quad \prod_{r=1}^k \left| \frac{\zeta(\frac{1}{2} + i\alpha_r^5)}{\zeta(\frac{1}{2} + i\alpha_r^4)} \right|^2 \sim \tan \left(\mu + \frac{U}{2} \right) \frac{\cos(\alpha_0^4)}{\sin(\alpha_0^5)}, \quad L \rightarrow \infty, \quad k = 1, \dots, k_0.$$

9.5. There is a kind of hierarchy in the class of sets of formulae we have obtained:

(a) First of all, we have two sets (see (9.2), (9.3))

$$(9.12) \quad \{[\zeta, Q^2, k; f_4]\}_{k=1}^{k_0}, \quad \{[\zeta, Q^2, k; f_5]\}_{k=1}^{k_0}$$

of the oscillating systems.

(b) Next, new set containing

$$(k_0)^2$$

elements of interactions

$$(9.13) \quad [\zeta, Q^2, k_1; f_4] \longleftrightarrow [\zeta, Q^2, k_2; f_5], \quad 1 \leq k_1, k_2 \leq k_0$$

between oscillating systems is generated by the sets (9.12) (see formula (9.6)).

(c) Now, by the subset

$$(9.14) \quad [\zeta, Q^2, k; f_4] \longleftrightarrow [\zeta, Q^2, k; f_5], \quad k = 1, \dots, k_0$$

of interactions (9.13) is generated the set of new type of factorization formulae (see (9.11))

$$\prod_{r=1}^k \left| \frac{\zeta(\frac{1}{2} + i\alpha_r^5)}{\zeta(\frac{1}{2} + i\alpha_r^4)} \right|^2 \sim \tan \left(\mu + \frac{U}{2} \right) \frac{\cos(\alpha_0^4)}{\sin(\alpha_0^5)}, \quad L \rightarrow \infty, \quad k = 1, \dots, k_0.$$

I would like to thank Michal Demetrian for his help with electronic version of this paper.

REFERENCES

- [1] J. Moser, 'Jacob's ladders and almost exact asymptotic representation of the Hardy-Littlewood integral', Math. Notes 88, (2010), 414-422, arXiv: 0901.3937.
- [2] J. Moser, 'Jacob's ladders, structure of the Hardy-Littlewood integral and some new class of nonlinear integral equations', Proc. Steklov Inst. 276 (2011), 208-221, arXiv: 1103.0359.
- [3] J. Moser, 'Jacob's ladders, reverse iterations and new infinite set of L_2 -orthogonal systems generated by the Riemann zeta-function', arXiv: 1402.2098.
- [4] J. Moser, 'Jacob's ladders, ζ -factorization and infinite set of metamorphoses of a multiform', arXiv: 1501.07705v2.
- [5] J. Moser, 'Jacob's ladders, Riemann's oscillators, quotient of the oscillating multiforms and set of metamorphoses of this system', arXiv: 1506.00442.
- [6] J. Moser, 'Jacob's ladders, factorization and metamorphoses as an appendix to the Riemann functional equation for $\zeta(s)$ on the critical line', arXiv: 1506.00442v1.
- [7] J. Moser, 'Jacob's ladders, \mathcal{Z}_{ζ, Q^2} -transformation of real elementary functions and telegraphic signals generated by the power functions', arXiv: 1602.04994.
- [8] C.L. Siegel, 'Über Riemann's Nachlass zur analytischen Zahlen-theorie : Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik', Abt. B, Studien 2 (1932), 45-80.

DEPARTMENT OF MATHEMATICAL ANALYSIS AND NUMERICAL MATHEMATICS, COMENIUS UNIVERSITY, MLYNSKA DOLINA M105, 842 48 BRATISLAVA, SLOVAKIA

E-mail address: jan.moser@fmph.uniba.sk